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The primitive L-pattern of angular momentum recoupling coefficients

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Labarthe's primitive L-patterns for the $3nj$-*symbols*, where $n = 3, 4, 5, 6, 7$, are reported. It is shown that, any L-patterns of the angular momentum recoupling coefficients can be expressed in terms of linear combinations of the primitive L-patterns and how the $3nj$-*symbols* can be calculated from the expressions presented here.

KEY WORDS: angular momentum, recoupling coefficients

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1. Introduction

When two dynamically independent systems, where their respective rotation operators commute with each other, and each one is characterized by their respective angular momentum eigenvalues and eigenstates, are coupled together, the matrix elements of the unitary transformation that connect the coupled and the two systems direct product representations, are the vector coupling or Clebsch–Gordan coefficients. For three independent systems coupled together, the coupling schemes and relevant transformations among them, give rise to the Racah coefficients or Wigner $6j$-symbols. As the number of independent systems

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coupled together increase, the relevant recoupling coefficients will be $9j$-, $12j$-, $15j$-, and $18j$-symbols for 4, 5, 6, 7 systems, respectively. As an example, the coupling of two spherical spin functions is a case of two-particle four-angular momenta coupling scheme, whereas the operators are their respective commuting orbital and spin angular momenta. Examples of higher order coupling schemes may be found in the L–S and the j–j coupling schemes in the theory of poly-electronic atoms.

The calculation of vector recoupling coefficients for more complex coupled systems finds applications in branches of physics and chemistry as the many particle theory develops but at the same time grows increasingly in complexity. For this reason the search for new theoretical approaches as well as alternative algorithms for their evaluation, have long encouraged research in the subject and still does.

Elegant formulae for the $3j$-, and $6j$-symbols have been obtained by Racah, Wigner and others and later by Labarthe [1–4]. For the $9j$-symbol case, the number of summations indexes increase rapidly and so far the simplest expression of the $9j$-symbol with three summation variables has been obtained by Jucys and co-worker [5].

Among the many contributions to the development of the theory, we can cite, in chronological order, the work by van der Waerden [6], Wigner [1], Racah [2], Sharp [3], Regge [7], Bargmann [8], Jucys and Bandzaitis [5], Varshelovich et al [9], Biedenharn and Louck [10], and Labarthe [4] and more recently Roothaan and Lai [11].

In this work, we have made use of the original ideas put forward by Labarthe and used his method to write the primitive L-patterns of 3nj-symbols. We also give the expressions for the $3nj$-symbols in terms of the coefficients of the linear combinations of the primitive L-patterns. The calculation of the $3j$-, $6j$- and $9j$-symbols are treated here in detail, while for the rest of the 3nj-symbols it is shown how they can be obtained from the L-patterns, since these are all explicitly listed here.

2. The algebra of primitive *L*-pattern

Let $\mathcal{E}$ denote the set of arrays L-pattern $[n \times m]$ formed of integers or half-integers that satisfy the triangle conditions for the $3nj$-symbols. Here n and m are the numbers of *rows* and columns of the array, respectively, and $m \geqslant n \geqslant 2$.

For example, $[2 \times 3]$ represents the array L-pattern $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $[3 \times 3]$ represents the array L-pattern $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and so on.

We assume that the following algebraic operations are satisfied by all the arrays of L-pattern. If $x,\ y \in \mathcal{E}$ and $\lambda \in \mathcal{N}$, where $\mathcal{N} \equiv \{0, 1, 2, \ldots\}$, then

$$x + y \in \mathcal{E},$$

and

$$\lambda(x + y) = \lambda x + \lambda y \in \mathcal{E}.$$

For example if

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}, \qquad y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix} \tag{1}$$

then we have

$$x + y = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} & x_{13} + y_{13} \\ x_{21} + y_{21} & x_{22} + y_{22} & x_{23} + y_{23} \end{bmatrix},$$

$$\lambda(x + y) = \begin{bmatrix} \lambda x_{11} + \lambda y_{11} & \lambda x_{12} + \lambda y_{12} & \lambda x_{13} + \lambda y_{13} \\ \lambda x_{21} + \lambda y_{21} & \lambda x_{22} + \lambda y_{22} & \lambda x_{23} + \lambda y_{23} \end{bmatrix}. \tag{2}$$

Let w be an L-pattern such that $w \in \mathcal{E}$. If w can be written as a linear combination of nonzero array L-patterns, such as $w = \lambda_1 x + \lambda_2 y$, where $x, y \in \mathcal{E}$ and $\lambda_1, \lambda_2 \in \mathcal{N}$, then w is said to be *reducible*.

A primitive L-pattern, e_λ, other than e_0, is defined as an L-pattern which cannot be expressed in terms of the sum of any other L-patterns.

For an specific L-pattern $[n \times m]$, the corresponding set of primitive L-patterns is completely defined and will be indicated by $(e_0, e_1, \ldots, e_{2^n})$.

For a given value of n, there are in general total number of 2^{n+1} primitive L-patterns. In the present work, the L-patterns considered correspond to $n = 2, 3, 4, 5, 6$. The value $n = 2$ refers to the well known $6j$ symbol, $n = 3$ refers to $9j$ symbol, etc.

We note here that the $3j$ symbol has some special property. The magnetic quantum number $m's(m_1 + m_2 + m_3 = 0)$ can be negative, but for the case of $6j$, $9j$, $12j$, $15j$, and $18j$ this is not considered because magnetic quantum numbers are not explicitly taken into account. Therefore, the total number of L-patterns for $3j$-symbols is less than for the $6j$-symbols.

3. Decomposition of 3*j*-symbols in terms of primitive *L*-pattern

The $3j$ symbols can be decomposed in *seven* (e_0 is included) primitive L-pattern, all in $\mathcal{E}$, which are depicted as follows:

$$e_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} s & s & 0 \\ s & -s & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & s & s \\ 0 & s & -s \end{bmatrix}, \quad e_3 = \begin{bmatrix} s & 0 & s \\ -s & 0 & s \end{bmatrix},$$
$$e_4 = \begin{bmatrix} s & 0 & s \\ s & 0 & -s \end{bmatrix}, \quad e_5 = \begin{bmatrix} s & s & 0 \\ -s & s & 0 \end{bmatrix}, \quad e_6 = \begin{bmatrix} 0 & s & s \\ 0 & -s & s \end{bmatrix}, \tag{3}$$

where $s = 1/2$ here and thereafter.

The values for the $3j$-symbols of the corresponding primitive L-pattern are 1 for e_0, $\sqrt{\frac{1}{2}}$ from e_1 to e_3 and $-1/\sqrt{2}$ from e_4 to e_6.

Any $x \in \mathcal{E}$ can be expressed in terms of primitive L-pattern as

$$x = \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix} = \sum_{k=1}^{6} \alpha_k e_k, \quad \text{where } \alpha_k \in \mathcal{N}. \tag{4}$$

Since the primitive L-patterns fulfill the relation

$$e_1 + e_2 + e_3 = e_4 + e_5 + e_6 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

it follows that the expansion of equation (4) is not unique. The general expression for the $3j$-symbols in terms of α_k is

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{1}{N_3} \sum_{\alpha_1,\ldots,\alpha_6} \frac{(-1)^{\alpha_4+\alpha_5+\alpha_6} \left(\sum_{i=1}^{6} \alpha_i + 1 \right)!}{\prod_{i=1}^{6} \alpha_i!}, \quad \text{where } 0 \leqslant \alpha_j \in \mathcal{N}. \tag{6}$$

Here, the sum runs over all the possible decompositions of $x \in \mathcal{E}$, in primitive L-patterns. In addition $m_1+m_2+m_3=0$ and $N_3=\sqrt{T_{j_1 j_2 j_3} T^{-}_{j_1 j_2 j_3, m_1 m_2 m_3} T^{+}_{j_1 j_2 j_3, m_1 m_2 m_3}}$, where [11]

$$\begin{aligned} T_{j_1 j_2 j_3} &= \frac{(j_1 + j_2 + j_3 + 1)!}{(-j_1 + j_2 + j_3)!(j_1 - j_2 + j_3)!(j_1 + j_2 - j_3)!}, \\ T^{-}_{j_1 j_2 j_3, m_1 m_2 m_3} &= \frac{(j_1 + j_2 + j_3 + 1)!}{(j_1 - m_1)!(j_2 - m_2)!(j_3 - m_3)!}, \\ T^{+}_{j_1 j_2 j_3, m_1 m_2 m_3} &= \frac{(j_1 + j_2 + j_3 + 1)!}{(j_1 + m_1)!(j_2 + m_2)!(j_3 + m_3)!}. \end{aligned} \tag{7}$$

Substituting the expression for the primitive L-patterns of equation (3) into the decomposition given in equation (4) and then equating the individual L-pattern elements on the left and on the right hand sides, we obtain a set of six linear equations on the α_k's. After solving these equations, the following α_k's are obtained.

$$\begin{aligned}
\alpha_1 &= j_1 + j_2 - j_3 - \alpha_5, \\
\alpha_2 &= j_2 + m_2 - \alpha_5, \\
\alpha_3 &= j_1 - m_1 - \alpha_5, \\
\alpha_4 &= j_3 - j_2 + m_1 + \alpha_5, \\
\alpha_6 &= j_3 - j_1 - m_2 + \alpha_5.
\end{aligned} \tag{8}$$

We see that the *six*-index sum in equation (6) is reduced to a *single* sum and the summation over α_5 must be restricted by the following conditions:

$$\max(j_1 - j_3 + m_2, j_2 - j_3 - m_1) \leqslant \alpha_5 \leqslant \min(j_1 + j_2 - j_3, j_2 + m_2, j_1 - m_1). \tag{9}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{1}{N_3} \sum_{\alpha_1,\ldots,\alpha_6} \frac{(-1)^{\alpha_4+\alpha_5+\alpha_6} \left(\sum_{i=1}^{6} \alpha_i + 1\right)!}{\prod_{i=1}^{6} \alpha_i!}, \quad 0 \leqslant \alpha_j \in \mathcal{N}. \tag{10}$$

It is easy to show that the above $3j$-*symbols* formula is exactly equal to Racah's expression [2]

$$\begin{aligned}
\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{1}{N_3} \sum_{\alpha_5} & \frac{(-1)^{j_1-j_2-m_3+\alpha_5}}{\alpha_5!(j_1+j_2-j_3-\alpha_5)!(j_2+m_2-\alpha_5)!} \\
& \times \frac{1}{(j_1-m_1-\alpha_5)!(j_3-j_2+m_1+\alpha_5)!(j_3-j_1-m_2+\alpha_5)}
\end{aligned} \tag{11}$$

4. Decomposition of 6*j*-symbols in terms of primitive *L*-pattern

For the case of $6j$-symbols (in $\mathcal{E}$), it is found that there are *eight* primitive L-pattern which are depicted as follows:

$$e_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} s & s & 0 \\ s & s & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & s & s \\ 0 & s & s \end{bmatrix}, \quad e_3 = \begin{bmatrix} s & 0 & s \\ s & 0 & s \end{bmatrix}, \tag{12}$$

$$e_4 = \begin{bmatrix} s & s & 0 \\ 0 & 0 & s \end{bmatrix}, \quad e_5 = \begin{bmatrix} 0 & s & s \\ s & 0 & 0 \end{bmatrix}, \quad e_6 = \begin{bmatrix} s & 0 & s \\ 0 & s & 0 \end{bmatrix}, \quad e_7 = \begin{bmatrix} 0 & 0 & 0 \\ s & s & s \end{bmatrix}. \tag{13}$$

The $6j$ values of corresponding primitive L-pattern are now: 1 for e_0, $1/2$ from e_1 to e_3, and $-1/\sqrt{2}$ from e_4 to e_7.

As before, any $x \in \mathcal{E}$ can be expressed in terms of primitive L-pattern as

$$x = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \sum_{k=1}^{7} \alpha_k e_k \quad \text{with } \alpha_k \in \mathcal{N}. \tag{14}$$

and also, since the primitive L-pattern satisfy now the relation

$$e_1 + e_2 + e_3 = e_4 + e_5 + e_6 + e_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{15}$$

the expansion of equation (13) is also not unique. The general expression [4, 11] for the $6j$ symbols in terms of α_k is

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \frac{1}{N_6} \sum_{\alpha_1,\ldots,\alpha_7} \frac{(-1)^{|\alpha|}(|\alpha|+1)!}{\prod_{j=1}^{7} \alpha_j!} \quad \text{with} \quad 0 \leqslant \alpha_j \in \mathcal{N}. \tag{16}$$

Where the sum runs over all the possible decompositions, in primitive L-pattern, of $x \in \mathcal{E}$. Here

$$N_6 = \sqrt{T_{abc} T_{aef} T_{bdf} T_{cde}}, \tag{17}$$

T_{abc} has been defined in equation (7), and

$$|\alpha| = \sum_{k=1}^{7} \alpha_k. \tag{18}$$

After solving equation (13) by using equation (12), then it is possible to express all $\alpha_k' s$ in terms of only α_7. These expression are as follows:

$$\begin{aligned}
\alpha_1 &= -c + d + e - \alpha_7, \\
\alpha_2 &= -a + e + f - \alpha_7, \\
\alpha_3 &= -b + d + f - \alpha_7, \\
\alpha_4 &= a + b - d - e + \alpha_7, \\
\alpha_5 &= b + c - e - f + \alpha_7, \\
\alpha_6 &= a + c - d - f + \alpha_7.
\end{aligned} \tag{19}$$

Thus it is shown that equation (15), which has *seven*-index sums has been reduced to *single* sum. The summation over α_7 must be restricted by the following condition:

$$0 \leqslant \alpha_7 \leqslant \min(-c + d + e, -a + e + f, -b + d + f). \tag{20}$$

Again, it is easy to show that the above $6j$ symbol is exactly equal to Racah's expression [3]

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \frac{1}{N_6} \sum_n \frac{(-1)^n (n+1)!}{(a+b+d+e-n)!(b+c+e+f-n)!(n-b-d-f)!} \times \frac{1}{(a+c+d+f-n)!(n-c-d-e)!(n-a-e-f)!(n-a-b-c)!}. \tag{21}$$

5. Decomposition of 9*j*-symbol in terms of primitive *L*-pattern

There are sixteen primitive L-pattern of $9j$-symbols in $\mathcal{E}$, which we list as follows:

$$
\begin{aligned}
&e_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&e_1 = \begin{bmatrix} s & s & 0 \\ s & s & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad
e_2 = \begin{bmatrix} s & 0 & s \\ s & 0 & s \\ 0 & 0 & 0 \end{bmatrix}, \quad
e_3 = \begin{bmatrix} 0 & s & s \\ 0 & s & s \\ 0 & 0 & 0 \end{bmatrix}, \\
&e_4 = \begin{bmatrix} s & s & 0 \\ 0 & 0 & 0 \\ s & s & 0 \end{bmatrix}, \quad
e_5 = \begin{bmatrix} s & 0 & s \\ 0 & 0 & 0 \\ s & 0 & s \end{bmatrix}, \quad
e_6 = \begin{bmatrix} 0 & s & s \\ 0 & 0 & 0 \\ 0 & s & s \end{bmatrix}, \\
&e_7 = \begin{bmatrix} 0 & 0 & 0 \\ s & s & 0 \\ s & s & 0 \end{bmatrix}, \quad
e_8 = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & s \\ s & 0 & s \end{bmatrix}, \quad
e_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & s & s \\ 0 & s & s \end{bmatrix}, \\
&e_{10} = \begin{bmatrix} 0 & s & s \\ s & 0 & s \\ s & s & 0 \end{bmatrix}, \quad
e_{11} = \begin{bmatrix} s & 0 & s \\ s & s & 0 \\ 0 & s & s \end{bmatrix}, \quad
e_{12} = \begin{bmatrix} s & s & 0 \\ 0 & s & s \\ s & 0 & s \end{bmatrix}, \\
&e_{13} = \begin{bmatrix} s & s & 0 \\ s & 0 & s \\ 0 & s & s \end{bmatrix}, \quad
e_{14} = \begin{bmatrix} 0 & s & s \\ s & s & 0 \\ s & 0 & s \end{bmatrix}, \quad
e_{15} = \begin{bmatrix} s & 0 & s \\ 0 & s & s \\ s & s & 0 \end{bmatrix},
\end{aligned}
\tag{22}
$$

where the $9j$ values of the corresponding primitive L-pattern are: 1 for e_0, (1/2) from e_1 to e_9, $-(1/4)$ from e_{10} to e_{12} and 1/4 from e_{13} to e_{15}, respectively.

As before, any given array $x \in \mathcal{E}$ of $9j$-symbol can be expanded in terms of primitive L-pattern as follows:

$$
x = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \sum_{k=1}^{15} \alpha_k e_k, \tag{23}
$$

where the $\alpha_k' s$ are non-negative integers.

Since the primitive L-patterns satisfy many relations similar to the following

$$
\begin{aligned}
&\sum_{i=1}^{9} e_i = \sum_{i=10}^{15} e_i = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \\
&e_1 + e_5 + e_9 + e_{10} = e_3 + e_4 + e_8 + e_{11} \\
&\qquad = \cdots = e_{13} + e_{14} + e_{15} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},
\end{aligned}
$$

the decomposition is again not unique. The general expression of the $9j$-symbol in terms of the $\alpha_k' s$ is now

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} = \frac{1}{N_9} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{15}} \frac{(-1)^{\alpha_{10}+\alpha_{11}+\alpha_{12}}(n+1)!}{\prod_{k=1}^{15} \alpha_k!}. \tag{24}$$

Following the same procedure described above, the 15-index summation can be reduced to 6 independent sums, from α_{10} to α_{15}, and where $\alpha_k \in \mathcal{N} \equiv \{0, 1, 2, \ldots\}$.

The relations among the $\alpha_k' s$ are now the following:

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(a+b-c+d+e-f-g-h+i+\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}+\alpha_{15}),\\
\alpha_2 &= \frac{1}{2}(a-b+c+d-e+f-g+h-i-\alpha_{10}-\alpha_{11}+\alpha_{12}-\alpha_{13}+\alpha_{14}-\alpha_{15}),\\
\alpha_3 &= \frac{1}{2}(-a+b+c-d+e+f+g-h-i-\alpha_{10}+\alpha_{11}-\alpha_{12}+\alpha_{13}-\alpha_{14}-\alpha_{15}),\\
\alpha_4 &= \frac{1}{2}(a+b-c-d-e+f+g+h-i-\alpha_{10}+\alpha_{11}-\alpha_{12}-\alpha_{13}+\alpha_{14}-\alpha_{15}),\\
\alpha_5 &= \frac{1}{2}(a-b+c-d+e-f+g-h+i+\alpha_{10}-\alpha_{11}-\alpha_{12}+\alpha_{13}-\alpha_{14}-\alpha_{15}),\\
\alpha_6 &= \frac{1}{2}(-a+b+c+d-e-f-g+h+i-\alpha_{10}-\alpha_{11}+\alpha_{12}-\alpha_{13}-\alpha_{14}+\alpha_{15}),\\
\alpha_7 &= \frac{1}{2}(-a-b+c+d+e-f+g+h-i-\alpha_{10}-\alpha_{11}+\alpha_{12}+\alpha_{13}-\alpha_{14}-\alpha_{15}),\\
\alpha_8 &= \frac{1}{2}(-a+b-c+d-e+f+g-h+i-\alpha_{10}+\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}+\alpha_{15}),\\
\alpha_9 &= \frac{1}{2}(a-b-c-d+e+f-g+h+i+\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}+\alpha_{14}-\alpha_{15})
\end{aligned} \tag{25}$$

and where

$$n = \sum_{i=1}^{15} \alpha_i = \frac{1}{2}(a+b+c+d+e+f+g+h+i \\ -\alpha_{10}-\alpha_{11}-\alpha_{12}-\alpha_{13}-\alpha_{14}-\alpha_{15}). \tag{26}$$

In equation (23) the sum over α_k must be restricted by the following conditions:

$$\begin{aligned}
&0 \leqslant \alpha_{13} + \alpha_{12} \leqslant \min(-c + f + i, a + b - c),\\
&0 \leqslant \alpha_{10} + \alpha_{14} \leqslant \min(-a + b + c, -a + d + g),\\
&0 \leqslant \alpha_{10} + \alpha_{15} \leqslant \min(c + f - i, g + h - i),\\
&0 \leqslant \alpha_{13} + \alpha_{11} \leqslant \min(a + d - g, -g + h + i),\\
&0 \leqslant \alpha_{15} + \alpha_{12} \leqslant \min(a - d + g, -d + e + f),\\
&0 \leqslant \alpha_{10} + \alpha_{13} \leqslant \min(d - e + f, b - e + h),\\
&0 \leqslant \alpha_{11} + \alpha_{14} \leqslant \min(c - f + i, d + e - f),\\
&0 \leqslant \alpha_{12} + \alpha_{14} \leqslant \min(g - h + i, b + e - h),\\
&0 \leqslant \alpha_{11} + \alpha_{15} \leqslant \min(a - b + c, -b + e + h).
\end{aligned} \tag{27}$$

It is seen that because in equation (23) the sum runs over *six* indices, numerical calculations using this expression are less efficient compared to $9j$-symbols computed as a sum of products of $6j$ symbols. That is

$$\begin{aligned}
\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} &= \frac{1}{\sqrt{T_{abc}T_{def}T_{ghi}T_{adg}T_{beh}T_{cfi}}}\\
&\times \sum_{u,v,w,x,y,z} \frac{(-1)^{u+v+w}(y+1)!}{u!v!w!x!z!(a-c+e-h+u-w+z)!(w-u-v-x-z+c+d-e+h-g)!}\\
&\times \frac{1}{(y+v+x-a-d-h-i)!(b-e+h-u-x-z)!(y+u+x-b-d-f-h)!}\\
&\times \frac{1}{(y+x+w-a-b-f-i)!(b+c+d+h+i-y-u-v-x-z)!}\\
&\times \frac{1}{(f+g-c-h-w+v+z)!(h+i-g-v-x-z)!(z-y-w+a+e+f+g)!}.
\end{aligned} \tag{28}$$

A MAPLE program, based on equation (24), has been written by one of us (STL). With the help of this program, the value of any given $9j$-symbol, and all possible decompositions, can be calculated.

When one of the argument of $9j$, for example, $i = 0$, equation (24) reduces to

$$\begin{aligned}
\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & 0 \end{Bmatrix} &= \frac{\delta(g,h)\delta(c,f)}{\sqrt{T_{abc}T_{def}T_{adg}T_{beh}T_{cf0}T_{gh0}}} \sum_s \frac{(-1)^s(b+c+d+h+1-s)!}{(a-c+e-h+s)!s!(d+h-a-s)!}\\
&\times \frac{1}{(c+d-e-s)!(c+d-a-s)!(b-e+h-s)!(a-b-d+e+s)!}.
\end{aligned} \tag{29}$$

6. Decomposition of 12*j*-symbol in terms of primitive *L*-pattern

In this case we find that there are thirty two (e_0 is included) primitive L-pattern of first kind $12j$-symbol [5,9] which are

$$
\begin{aligned}
&e_1=\begin{bmatrix}0&0&s&0\\0&s&s&0\\0&0&s&0\end{bmatrix},\quad e_2=\begin{bmatrix}s&0&0&0\\s&0&0&s\\s&0&0&0\end{bmatrix},\quad e_3=\begin{bmatrix}0&0&0&s\\0&0&s&s\\0&0&0&s\end{bmatrix},\quad e_4=\begin{bmatrix}0&s&0&0\\s&s&0&0\\0&s&0&0\end{bmatrix},\\
&e_5=\begin{bmatrix}0&0&s&s\\0&s&0&s\\0&0&s&s\end{bmatrix},\quad e_6=\begin{bmatrix}s&0&0&s\\s&0&s&0\\s&0&0&s\end{bmatrix},\quad e_7=\begin{bmatrix}0&s&s&0\\s&0&s&0\\0&s&s&0\end{bmatrix},\quad e_8=\begin{bmatrix}s&s&0&0\\0&s&0&s\\s&s&0&0\end{bmatrix},\\
&e_9=\begin{bmatrix}0&s&0&s\\s&s&s&s\\0&s&0&s\end{bmatrix},\ e_{10}=\begin{bmatrix}s&0&s&0\\s&s&s&s\\s&0&s&0\end{bmatrix},\ e_{11}=\begin{bmatrix}s&s&s&0\\0&0&s&s\\s&s&s&0\end{bmatrix},\ e_{12}=\begin{bmatrix}0&s&s&s\\s&0&0&s\\0&s&s&s\end{bmatrix},\\
&e_{13}=\begin{bmatrix}s&0&s&s\\s&s&0&0\\s&0&s&s\end{bmatrix},\ e_{14}=\begin{bmatrix}s&s&0&s\\0&s&s&0\\s&s&0&s\end{bmatrix},\ e_{15}=\begin{bmatrix}s&s&s&s\\0&0&0&0\\s&s&s&s\end{bmatrix},\ e_{16}=\begin{bmatrix}s&s&s&0\\0&0&s&0\\0&0&0&s\end{bmatrix},\\
&e_{17}=\begin{bmatrix}s&s&s&s\\0&0&0&s\\0&0&0&0\end{bmatrix},\ e_{18}=\begin{bmatrix}0&0&0&s\\0&0&s&0\\s&s&s&0\end{bmatrix},\ e_{19}=\begin{bmatrix}0&0&0&0\\0&0&0&s\\s&s&s&s\end{bmatrix},\ e_{20}=\begin{bmatrix}s&s&0&0\\0&s&0&0\\0&0&s&s\end{bmatrix},\\
&e_{21}=\begin{bmatrix}s&0&0&0\\s&0&0&0\\0&s&s&s\end{bmatrix},\ e_{22}=\begin{bmatrix}0&0&s&s\\0&s&0&0\\s&s&0&0\end{bmatrix},\ e_{23}=\begin{bmatrix}0&s&s&s\\s&0&0&0\\s&0&0&0\end{bmatrix},\ e_{24}=\begin{bmatrix}s&0&0&s\\s&0&s&s\\0&s&s&0\end{bmatrix},\\
&e_{25}=\begin{bmatrix}0&s&0&0\\s&s&0&s\\s&0&s&s\end{bmatrix},\ e_{26}=\begin{bmatrix}0&s&0&s\\s&s&s&0\\s&0&s&0\end{bmatrix},\ e_{27}=\begin{bmatrix}s&0&s&0\\s&s&s&0\\0&s&0&s\end{bmatrix},\ e_{28}=\begin{bmatrix}s&0&s&s\\s&s&0&s\\0&s&0&0\end{bmatrix},\\
&e_{29}=\begin{bmatrix}0&0&s&0\\0&s&s&s\\s&s&0&s\end{bmatrix},\ e_{30}=\begin{bmatrix}0&s&s&0\\s&0&s&s\\s&0&0&s\end{bmatrix},\ e_{31}=\begin{bmatrix}s&s&0&s\\0&s&s&s\\0&0&s&0\end{bmatrix},
\end{aligned}
\tag{30}
$$

where the values of the $12j$-symbols corresponding to the above primitive L-pattern are: (1/2) from e_1 to e_4, (1/4) from e_5 to e_{10}, (1/8) from e_{11} to e_{14}, $-(1/8)$ from e_{15}, $(\sqrt{2}/4)$ from e_{16} to e_{23}, and $-(\sqrt{2}/8)$ from e_{24} to e_{31}, respectively.

As before, any given array $x \in \mathcal{E}$ of $12j$-symbol can be expanded in terms of primitive L-pattern in the following way:

$$
x=\begin{bmatrix}a_1&a_2&a_3&a_4\\b_{12}&b_{23}&b_{34}&b_{41}\\c_1&c_2&c_3&c_4\end{bmatrix}=\sum_{\ell=1}^{31}\alpha_\ell e_\ell, \tag{31}
$$

where the α_ℓ's are non-negative integers. Since the primitive L-patterns satisfy relations of the type:

$$\sum_{i=1}^{15} e_i = \sum_{i=16}^{31} e_i = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}, \tag{32}$$

the decomposition is also not unique.
The general expression of the $12j$-symbol in terms of α_ℓ is

$$\begin{Bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_{12} & b_{23} & b_{34} & b_{41} \\ c_1 & c_2 & c_3 & c_4 \end{Bmatrix} = \frac{1}{N_{12}} \sum_{\{\alpha\}} \frac{(-1)^{\sum_{i=1}^{31} \alpha_i} (|\alpha| + 1)!}{\prod_{p=1}^{31} \alpha_p!}. \tag{33}$$

In equation (32) $|\alpha|$ and $\{\alpha\}$signify

$$|\alpha| = \sum_{\ell=1}^{31} \alpha_\ell \quad \text{and} \quad \{\alpha\} \equiv \{\alpha_1, \ldots, \alpha_{31}\},$$

respectively, and

$$N_{12} = \sqrt{T_{a_1 a_2 b_{12}} T_{a_2 a_3 b_{23}} \ldots T_{c_4 b_{41} a_1}}.$$

7. Decomposition of 15*j*-symbol in terms of primitive *L*-pattern

Below, we list the sixty four (e_0 is included) primitive L-pattern arising of first kind $15j$-symbol [5]

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 0 & s & 0 & 0 \\ 0 & s & s & 0 & 0 \\ 0 & 0 & s & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & s & 0 & 0 & 0 \\ s & s & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & s & s \\ 0 & 0 & 0 & 0 & s \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 & 0 & s & 0 \\ 0 & 0 & s & s & 0 \\ 0 & 0 & 0 & s & 0 \end{bmatrix}, \quad e_5 = \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & s \\ s & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \\ s & s & s & s & s \end{bmatrix}, \quad e_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & s & 0 \\ s & s & s & s & 0 \end{bmatrix}, \quad e_8 = \begin{bmatrix} 0 & 0 & 0 & s & s \\ 0 & 0 & s & 0 & 0 \\ s & s & s & 0 & 0 \end{bmatrix},$$

$$e_9 = \begin{bmatrix} 0&0&0&s&s\\ 0&0&s&0&s\\ 0&0&0&s&s \end{bmatrix},\quad e_{10} = \begin{bmatrix} 0&0&s&s&0\\ 0&s&0&s&0\\ 0&0&s&s&0 \end{bmatrix},\quad e_{11} = \begin{bmatrix} 0&0&s&s&s\\ 0&s&0&0&0\\ s&s&0&0&0 \end{bmatrix},$$

$$e_{12} = \begin{bmatrix} s&0&0&0&0\\ s&0&0&0&0\\ 0&s&s&s&s \end{bmatrix},\quad e_{13} = \begin{bmatrix} 0&s&s&s&s\\ s&0&0&0&0\\ s&0&0&0&0 \end{bmatrix},\quad e_{14} = \begin{bmatrix} 0&s&s&0&0\\ s&0&s&0&0\\ 0&s&s&0&0 \end{bmatrix},$$

$$e_{15} = \begin{bmatrix} s&0&0&0&s\\ s&0&0&s&0\\ s&0&0&0&s \end{bmatrix},\quad e_{16} = \begin{bmatrix} s&s&0&0&0\\ 0&s&0&0&0\\ 0&0&s&s&s \end{bmatrix},\quad e_{17} = \begin{bmatrix} s&s&0&0&0\\ 0&s&0&0&s\\ s&s&0&0&0 \end{bmatrix},$$

$$e_{18} = \begin{bmatrix} s&s&s&0&0\\ 0&0&s&0&0\\ 0&0&0&s&s \end{bmatrix},\quad e_{19} = \begin{bmatrix} s&s&s&s&0\\ 0&0&0&s&0\\ 0&0&0&0&s \end{bmatrix},\quad e_{20} = \begin{bmatrix} s&s&s&s&s\\ 0&0&0&0&s\\ 0&0&0&0&0 \end{bmatrix},$$

$$e_{21} = \begin{bmatrix} 0&s&0&s&0\\ s&s&s&s&0\\ 0&s&0&s&0 \end{bmatrix},\quad e_{22} = \begin{bmatrix} 0&0&s&0&s\\ 0&s&s&s&s\\ 0&0&s&0&s \end{bmatrix},\quad e_{23} = \begin{bmatrix} 0&s&0&0&s\\ s&s&0&s&s\\ 0&s&0&0&s \end{bmatrix},$$

$$e_{24} = \begin{bmatrix} s&0&s&0&0\\ s&s&s&0&s\\ s&0&s&0&0 \end{bmatrix},\quad e_{25} = \begin{bmatrix} s&0&0&s&0\\ s&0&s&s&s\\ s&0&0&s&0 \end{bmatrix},\quad e_{26} = \begin{bmatrix} s&s&0&0&s\\ 0&s&0&s&0\\ s&s&0&0&s \end{bmatrix},$$

$$e_{27} = \begin{bmatrix} s&0&0&s&s\\ s&0&s&0&0\\ s&0&0&s&s \end{bmatrix},\quad e_{28} = \begin{bmatrix} s&s&s&0&0\\ 0&0&s&0&s\\ s&s&s&0&0 \end{bmatrix},\quad e_{29} = \begin{bmatrix} 0&s&s&s&0\\ s&0&0&s&0\\ 0&s&s&s&0 \end{bmatrix},$$

$$e_{30} = \begin{bmatrix} 0&0&s&s&s\\ 0&s&0&0&s\\ 0&0&s&s&s \end{bmatrix},\quad e_{31} = \begin{bmatrix} 0&0&s&0&0\\ 0&s&s&0&s\\ s&s&0&s&s \end{bmatrix},\quad e_{32} = \begin{bmatrix} 0&s&0&0&s\\ s&s&0&s&0\\ s&0&s&s&0 \end{bmatrix},$$

$$e_{33} = \begin{bmatrix} 0&0&s&0&s\\ 0&s&s&s&0\\ s&s&0&s&0 \end{bmatrix},\quad e_{34} = \begin{bmatrix} 0&0&s&s&0\\ 0&s&0&s&s\\ s&s&0&0&s \end{bmatrix},\quad e_{35} = \begin{bmatrix} 0&s&0&s&s\\ s&s&s&0&0\\ s&0&s&0&0 \end{bmatrix},$$

$$e_{36} = \begin{bmatrix} 0&s&s&0&0\\ s&0&s&0&s\\ s&0&0&s&s \end{bmatrix},\quad e_{37} = \begin{bmatrix} 0&0&0&s&0\\ 0&0&s&s&s\\ s&s&s&0&s \end{bmatrix},\quad e_{38} = \begin{bmatrix} 0&s&0&0&0\\ s&s&0&0&s\\ s&0&s&s&s \end{bmatrix},$$

$$e_{39} = \begin{bmatrix} 0&s&s&0&s\\ s&0&s&s&0\\ s&0&0&s&0 \end{bmatrix},\quad e_{40} = \begin{bmatrix} 0&s&s&s&0\\ s&0&0&s&s\\ s&0&0&0&s \end{bmatrix},\quad e_{41} = \begin{bmatrix} s&0&0&0&s\\ s&0&0&s&s\\ 0&s&s&s&0 \end{bmatrix},$$

$$e_{42} = \begin{bmatrix} s&0&0&s&0\\ s&0&s&s&0\\ 0&s&s&0&s \end{bmatrix},\quad e_{43} = \begin{bmatrix} s&0&0&s&s\\ s&0&s&0&s\\ 0&s&s&0&0 \end{bmatrix},\quad e_{44} = \begin{bmatrix} s&s&0&0&s\\ 0&s&0&s&s\\ 0&0&s&s&0 \end{bmatrix},$$

$$e_{45} = \begin{bmatrix} s & 0 & s & 0 & 0 \\ s & s & s & 0 & 0 \\ 0 & s & 0 & s & s \end{bmatrix}, \quad e_{46} = \begin{bmatrix} s & s & 0 & s & 0 \\ 0 & s & s & s & 0 \\ 0 & 0 & s & 0 & s \end{bmatrix}, \quad e_{47} = \begin{bmatrix} s & s & 0 & s & s \\ 0 & s & s & 0 & s \\ 0 & 0 & s & 0 & 0 \end{bmatrix},$$

$$e_{48} = \begin{bmatrix} s & 0 & s & s & 0 \\ s & s & 0 & s & 0 \\ 0 & s & 0 & 0 & s \end{bmatrix}, \quad e_{49} = \begin{bmatrix} s & 0 & s & s & s \\ s & s & 0 & 0 & s \\ 0 & s & 0 & 0 & 0 \end{bmatrix}, \quad e_{50} = \begin{bmatrix} s & s & s & 0 & s \\ 0 & 0 & s & s & s \\ 0 & 0 & 0 & s & 0 \end{bmatrix},$$

$$e_{51} = \begin{bmatrix} s & 0 & s & 0 & s \\ s & s & s & s & 0 \\ s & 0 & s & 0 & s \end{bmatrix}, \quad e_{52} = \begin{bmatrix} s & 0 & s & s & 0 \\ s & s & 0 & s & s \\ s & 0 & s & s & 0 \end{bmatrix}, \quad e_{53} = \begin{bmatrix} s & s & 0 & s & 0 \\ 0 & s & s & s & s \\ s & s & 0 & s & 0 \end{bmatrix},$$

$$e_{54} = \begin{bmatrix} 0 & s & s & 0 & s \\ s & 0 & s & s & s \\ 0 & s & s & 0 & s \end{bmatrix}, \quad e_{55} = \begin{bmatrix} 0 & s & 0 & s & s \\ s & s & s & 0 & s \\ 0 & s & 0 & s & s \end{bmatrix}, \quad e_{56} = \begin{bmatrix} 0 & s & 0 & s & 0 \\ s & s & s & s & s \\ s & 0 & s & 0 & s \end{bmatrix},$$

$$e_{57} = \begin{bmatrix} 0 & s & s & s & s \\ s & 0 & 0 & 0 & s \\ 0 & s & s & s & s \end{bmatrix}, \quad e_{58} = \begin{bmatrix} s & s & 0 & s & s \\ 0 & s & s & 0 & 0 \\ s & s & 0 & s & s \end{bmatrix}, \quad e_{59} = \begin{bmatrix} s & 0 & s & 0 & s \\ s & s & s & s & s \\ 0 & s & 0 & s & 0 \end{bmatrix},$$

$$e_{60} = \begin{bmatrix} s & 0 & s & s & s \\ s & s & 0 & 0 & 0 \\ s & 0 & s & s & s \end{bmatrix}, \quad e_{61} = \begin{bmatrix} s & s & s & 0 & s \\ 0 & 0 & s & s & 0 \\ s & s & s & 0 & s \end{bmatrix}, \quad e_{62} = \begin{bmatrix} s & s & s & s & 0 \\ 0 & 0 & 0 & s & s \\ s & s & s & s & 0 \end{bmatrix},$$

$$e_{63} = \begin{bmatrix} s & s & s & s & s \\ 0 & 0 & 0 & 0 & 0 \\ s & s & s & s & s \end{bmatrix}, \tag{34}$$

where the values of the $15j$-symbols corresponding to the above primitive L-pattern are: 1 for e_0, (1/2) from e_1 to e_5, (1/4) from e_6 to e_{25}, (1/8) from e_{26} to e_{30}, $-(1/8)$ from e_{31} to e_{50}, 1/8 from e_{51} to e_{55}, (1/16) from e_{56} to e_{62}, and $-(1/16)$ for e_{63}, respectively.

The expansion, in terms of primitive L-pattern, of any given $x \in \mathcal{E}$ of $15j$-symbol is as follows:

$$x = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_{12} & b_{23} & b_{34} & b_{45} & b_{51} \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix} = \sum_{k=1}^{63} \alpha_k e_k, \tag{35}$$

where, as before, α_k's are non-negative integers.

The decomposition is not unique since the primitive L-patterns again satisfy many relations of the type

$$\sum_{i=1}^{5} e_i + \sum_{i=21}^{50} e_i = \sum_{i=6}^{20} e_i + \sum_{i=51}^{63} e_i = \begin{bmatrix} 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \end{bmatrix}. \tag{36}$$

The general expression of the $15j$-symbol in terms of α_k is

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_{12} & b_{23} & b_{34} & b_{45} & b_{51} \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix}\right\} = \frac{1}{N_{15}} \sum_{\{\alpha\}} \frac{(-1)^{\sum_{i=31}^{50}\alpha_i}(-1)^{\alpha_{63}}(n+1)!}{\prod_{k=1}^{63}\alpha_k!} \tag{37}$$

in which the following definitions apply:

$$n = \sum_{i=1}^{63}\alpha_i, \qquad \{\alpha\} \equiv \{\alpha_1, \dots, \alpha_{63}\}$$

and

$$N_{15} = \sqrt{T_{a_1a_2b_{12}}T_{a_2a_3b_{23}}\dots T_{c_5b_{51}a_1}}.$$

8. Decomposition of 18*j*-symbol in terms of primitive *L*-pattern

There are 128 (e_0 is included) primitive L-pattern of first kind $18j$-symbol [7] Nevertheless, for the sake of simplicity, we will only show those primitive L-patterns that give rise to different $18j$-symbols values.[1] These primitive L-patterns are the following:

$$e_0 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0&0&0&0&0&s\\ 0&0&0&0&s&s\\ 0&0&0&0&0&s \end{bmatrix}, \quad e_7 = \begin{bmatrix} 0&0&0&0&s&s\\ 0&0&0&s&0&s\\ 0&0&0&0&s&s \end{bmatrix},$$
$$e_{22} = \begin{bmatrix} 0&0&0&s&s&s\\ 0&0&s&0&0&s\\ 0&0&0&s&s&s \end{bmatrix}, \quad e_{42} = \begin{bmatrix} 0&s&s&s&s&0\\ s&0&0&0&s&0\\ 0&s&s&s&s&0 \end{bmatrix}, \quad e_{57} = \begin{bmatrix} 0&s&s&s&s&s\\ s&0&0&0&0&s\\ 0&s&s&s&s&s \end{bmatrix},$$
$$e_{63} = \begin{bmatrix} s&s&s&s&s&s\\ 0&0&0&0&0&0\\ s&s&s&s&s&s \end{bmatrix}, \quad e_{64} = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&s\\ s&s&s&s&s&s \end{bmatrix}, \quad e_{76} = \begin{bmatrix} 0&0&0&s&0&0\\ 0&0&s&s&0&s\\ s&s&s&0&s&s \end{bmatrix},$$
$$e_{116} = \begin{bmatrix} 0&0&s&0&s&0\\ 0&s&s&s&s&s\\ s&s&0&s&0&s \end{bmatrix}. \tag{38}$$

The values of the corresponding $18j$-symbols are: 1 for e_0, (1/2) from e_1 to e_6, (1/4) from e_7 to e_{21}, (1/8) from e_{22} to e_{41}, (1/16) from e_{42} to e_{56}, (1/32) from e_{57} to e_{62}, and $-(1/32)$ for e_{63},($\sqrt{2}/8$) from e_{64} to e_{75}, $-(\sqrt{2}/16)$ from e_{76} to e_{115}, ($\sqrt{2}/32$) from e_{116} to e_{127}, respectively.

Any given array x(in $\mathcal{E}$) of $18j$-symbol can be expanded in terms of primitive L-pattern as follows:

[1]The full set of 128 L-patterns may be obtained from the authors upon request.

$$x = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_{12} & b_{23} & b_{34} & b_{45} & b_{56} & b_{61} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} = \sum_{k=1}^{127} \alpha_k e_k, \tag{39}$$

where α_k's are non-negative integers.

Due to the many relations of the type

$$\sum_{i=1}^{63} e_i = \sum_{i=64}^{127} e_i = \begin{bmatrix} 16 & 16 & 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 & 16 & 16 \\ 16 & 16 & 16 & 16 & 16 & 16 \end{bmatrix} \tag{40}$$

fulfilled by the primitive L-pattern the decomposition is not unique. The general expression of the $18j$-symbol in terms of α_k will be

$$\begin{Bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_{12} & b_{23} & b_{34} & b_{45} & b_{56} & b_{61} \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{Bmatrix} = \frac{1}{N_{18}} \sum_{\{\alpha\}} \frac{(-1)^{\sum_{i=1}^{127} \alpha_i} (|\alpha| + 1)!}{\prod_{\ell=1}^{127} \alpha_\ell !}, \tag{41}$$

with the definitions:

$$|\alpha| = \sum_{i=1}^{127} \alpha_i, \qquad \{\alpha\} \equiv \{\alpha_1, \ldots, \alpha_{127}\}$$

and

$$N_{18} = \sqrt{T_{a_1 a_2 b_{12}} T_{a_2 a_3 b_{23}} \ldots T_{c_6 b_{61} a_1}}.$$

9. Concluding remarks

In this article we have intended to present all the basic L-patterns from which, in principle, the $3nj$-symbols can be theoretically calculated since the L-patterns are primitive to all of them. In this manner, we wanted to explore an alternative view to simplify the problem of computing the higher order angular momentum recoupling coefficients with a different simple technique. Further exploratory work is being carried out, aimed to reduce the number of summation indexes in the expressions here presented.

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